UNIAXIAL STRESS-STRAIN

Stress-Strain Curve for Mild Steel

The slope of the linear portion of the curve equals the modulus of elasticity.

**DEFINITIONS**

Engineering Strain

\[ \varepsilon = \frac{\Delta L}{L_0}, \]

- \( \varepsilon \) = engineering strain (units per unit),
- \( \Delta L \) = change in length (units) of member,
- \( L_0 \) = original length (units) of member.

Percent Elongation

\[ \% \text{ Elongation} = \left( \frac{\Delta L}{L_0} \right) \times 100 \]

Percent Reduction in Area (RA)

The % reduction in area from initial area, \( A_i \), to final area, \( A_f \), is:

\[ \% \text{RA} = \left( \frac{A_i - A_f}{A_i} \right) \times 100 \]

True Stress is load divided by actual cross-sectional area.

Shear Stress-Strain

\[ \gamma = \frac{\tau}{G} \]

- \( \gamma \) = shear strain,
- \( \tau \) = shear stress,
- \( G \) = shear modulus (constant in linear force-deformation relationship).

\[ G = \frac{E}{2(1 + v)} \]

- \( E \) = modulus of elasticity
- \( v \) = Poisson's ratio, and
- \( v = \delta_{l} / \varepsilon_{l} \) = (lateral strain)/(longitudinal strain).

Uniaxial Loading and Deformation

\[ \sigma = \frac{P}{A}, \]

- \( \sigma \) = stress on the cross section,
- \( P \) = loading, and
- \( A \) = cross-sectional area.

\[ \varepsilon = \frac{\delta}{L}, \]

- \( \varepsilon \) = strain,
- \( \delta \) = elastic longitudinal deformation and
- \( L \) = length of member.

\[ E = \frac{P}{\delta L} \]

\[ \delta = \frac{PL}{AE} \]

**THERMAL DEFORMATIONS**

\[ \delta_t = \alpha L (T - T_0), \]

- \( \delta_t \) = deformation caused by a change in temperature,
- \( \alpha \) = temperature coefficient of expansion,
- \( L \) = length of member,
- \( T \) = final temperature, and
- \( T_0 \) = initial temperature.

**CYLINDRICAL PRESSURE VESSEL**

Cylindrical Pressure Vessel

For internal pressure only, the stresses at the inside wall are:

\[ \sigma_t = P_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad 0 > \sigma_t > -P_i \]

For external pressure only, the stresses at the outside wall are:

\[ \sigma_t = -P_o \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} \quad \text{and} \quad 0 > \sigma_t > -P_o \]

- \( \sigma_t \) = tangential (hoop) stress,
- \( \sigma_r \) = radial stress,
- \( P_i \) = internal pressure,
- \( P_o \) = external pressure,
- \( r_i \) = inside radius, and
- \( r_o \) = outside radius.

For vessels with end caps, the axial stress is:

\[ \sigma_a = P_i \frac{r_i^2}{r_o^2 - r_i^2} \]

These are principal stresses.

When the thickness of the cylinder wall is about one-tenth or less, of inside radius, the cylinder can be considered as thin-walled. In which case, the internal pressure is resisted by the hoop stress and the axial stress.

\[ \sigma_t = \frac{P_r}{t} \quad \text{and} \quad \sigma_a = \frac{P_r}{2t} \]

where \( t \) = wall thickness.

**STRESS AND STRAIN**

**Principal Stresses**

For the special case of a two-dimensional stress state, the equations for principal stress reduce to

\[ \sigma_a, \sigma_b = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \]

\[ \sigma_c = 0 \]

The two nonzero values calculated from this equation are temporarily labeled \( \sigma_a \) and \( \sigma_b \) and the third value \( \sigma_c \) is always zero in this case. Depending on their values, the three roots are then labeled according to the convention: algebraically largest = \( \sigma_1 \), algebraically smallest = \( \sigma_3 \), other = \( \sigma_2 \). A typical 2D stress element is shown below with all indicated components shown in their positive sense.

![Mohr's Circle - Stress, 2D](image)

The circle drawn with the center on the normal stress (horizontal) axis with center, C, and radius, R, where

\[ C = \frac{\sigma_x + \sigma_y}{2}, \quad R = \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \]

The two nonzero principal stresses are then:

\[ \sigma_a = C + R \]
\[ \sigma_b = C - R \]

The maximum in-plane shear stress is \( \tau_{in} = R \). However, the maximum shear stress considering three dimensions is always

\[ \tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \]

**Hooke’s Law**

Three-dimensional case:

\[ \epsilon_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) \]
\[ \gamma_{xy} = \tau_{xy}/G \]
\[ \epsilon_y = \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) \]
\[ \gamma_{yz} = \tau_{yz}/G \]
\[ \epsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)) \]
\[ \gamma_{zx} = \tau_{zx}/G \]

Plane stress case (\( \sigma_z = 0 \)):

\[ \epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \]
\[ \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \]
\[ \epsilon_z = -\frac{1}{E}(\nu\sigma_x + \nu\sigma_y) \]
\[ \gamma_{xy} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \]

Uniaxial case (\( \sigma_y = \sigma_z = 0 \)): \( \sigma_x = E\epsilon_x \) or \( \sigma = E\epsilon \), where
\( \epsilon_x, \epsilon_y, \epsilon_z = \) normal strain,
\( \sigma_{xx}, \sigma_{yy}, \sigma_{zz} = \) normal stress,
\( \gamma_{xy}, \gamma_{yz}, \gamma_{zx} = \) shear strain,
\( \tau_{xy}, \tau_{yz}, \tau_{zx} = \) shear stress,
\( E = \) modulus of elasticity,
\( G = \) shear modulus, and
\( \nu = \) Poisson’s ratio.


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**MECHANICS OF MATERIALS (continued)**
STATIC LOADING FAILURE THEORIES

Brittle Materials

Maximum-Normal-Stress Theory
The maximum-normal-stress theory states that failure occurs when one of the three principal stresses equals the strength of the material. If \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \), then the theory predicts that failure occurs whenever \( \sigma_1 \geq S_{ut} \) or \( \sigma_3 \leq -S_{uc} \) where \( S_{ut} \) and \( S_{uc} \) are the tensile and compressive strengths, respectively.

Coulomb-Mohr Theory
The Coulomb-Mohr theory is based upon the results of tensile and compression tests. On the \( \sigma, \tau \) coordinate system, one circle is plotted for \( S_{ut} \) and one for \( S_{uc} \). As shown in the figure, lines are then drawn tangent to these circles. The Coulomb-Mohr theory then states that fracture will occur for any stress situation that produces a circle that is either tangent to or crosses the envelope defined by the lines tangent to the \( S_{ut} \) and \( S_{uc} \) circles.

If \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \) and \( \sigma_3 < 0 \), then the theory predicts that yielding will occur whenever

\[
\frac{\sigma_1 - \sigma_3}{S_{ut}} \geq 1
\]

Ductile Materials

Maximum-Shear-Stress Theory
The maximum-shear-stress theory states that yielding begins when the maximum shear stress equals the maximum shear stress in a tension-test specimen of the same material when that specimen begins to yield. If \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \), then the theory predicts that yielding will occur whenever \( \tau_{\text{max}} \geq S_y/2 \) where \( S_y \) is the yield strength.

Distortion-Energy Theory
The distortion-energy theory states that yielding begins whenever the distortion energy in a unit volume equals the distortion energy in the same volume when uniaxially stressed to the yield strength. The theory predicts that yielding will occur whenever

\[
\frac{1}{2} \left( (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \right) \geq S_y
\]

The term on the left side of the inequality is known as the effective or Von Mises stress. For a biaxial stress state the effective stress becomes

\[
\sigma' = \left( \sigma_A^2 - \sigma_A \sigma_B + \sigma_B^2 \right)^{1/2}
\]

or

\[
\sigma' = \left( \sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3 \tau_{xy}^2 \right)^{1/2}
\]

where \( \sigma_A \) and \( \sigma_B \) are the two nonzero principal stresses and \( \sigma_x, \sigma_y, \tau_{xy} \) are the stresses in orthogonal directions.

VARIABLE LOADING FAILURE THEORIES

Modified Goodman Theory: The modified Goodman criterion states that a fatigue failure will occur whenever

\[
\frac{\sigma_a + \sigma_m}{S_e} + \frac{\sigma_{\text{max}}}{S_y} \geq 1, \quad \sigma_m \geq 0
\]

Soderberg Theory: The Soderberg theory states that a fatigue failure will occur whenever

\[
\frac{\sigma_a + \sigma_m}{S_y} \geq 1, \quad \sigma_m \geq 0
\]

Endurance Limit for Steels: When test data is unavailable, the endurance limit for steels may be estimated as

\[
S_{\text{c}} = \begin{cases} 
0.5 S_{ut}, & S_{ut} \leq 1,400 \text{ MPa} \\
700 \text{ MPa}, & S_{ut} > 1,400 \text{ MPa}
\end{cases}
\]
Endurance Limit Modifying Factors: Endurance limit modifying factors are used to account for the differences between the endurance limit as determined from a rotating beam test, \( S'_e \), and that which would result in the real part, \( S_e \).

\[ S_e = k_a k_b k_c k_d k_e S'_e \]

where

Surface Factor, \( k_a = aS^{b_h} \)

<table>
<thead>
<tr>
<th>Surface Finish</th>
<th>Factor ( a ) (ksi)</th>
<th>Exponent ( b )</th>
<th>Factor ( a ) (MPa)</th>
<th>Exponent ( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground</td>
<td>1.34</td>
<td>–0.085</td>
<td>1.58</td>
<td>–0.085</td>
</tr>
<tr>
<td>Machined or CD</td>
<td>2.70</td>
<td>–0.265</td>
<td>4.51</td>
<td>–0.265</td>
</tr>
<tr>
<td>Hot rolled</td>
<td>14.4</td>
<td>–0.718</td>
<td>57.7</td>
<td>–0.718</td>
</tr>
<tr>
<td>As forged</td>
<td>39.9</td>
<td>–0.995</td>
<td>272.0</td>
<td>–0.995</td>
</tr>
</tbody>
</table>

Size Factor, \( k_b \):

For bending and torsion:

\[ k_b = \begin{cases} 1 & d \leq 8 \text{ mm}; \\ 1.189d^{-0.097} & 8 \text{ mm} \leq d \leq 250 \text{ mm}; \\ 0.6 \leq k_b \leq 0.75 & d > 250 \text{ mm}; \end{cases} \]

Load Factor, \( k_c \):

\[ k_c = \begin{cases} 0.923 & \text{axial loading, } S_{ut} \leq 1,520 \text{ MPa} \\ 1 & \text{axial loading, } S_{ut} > 1,520 \text{ MPa} \\ 1 & \text{bending} \end{cases} \]

Temperature Factor, \( k_d \):

\[ k_d = \begin{cases} 1 & T \leq 450^\circ \text{C} \end{cases} \]

Miscellaneous Effects Factor, \( k_e \): Used to account for strength reduction effects such as corrosion, plating, and residual stresses. In the absence of known effects, use \( k_e = 1 \).

\[ T/\phi \text{ gives the twisting moment per radian of twist. This is called the torsional stiffness and is often denoted by the symbol } k \text{ or } c. \]

For Hollow, Thin-Walled Shafts

\[ \tau = \frac{T}{2Am^2} \]

\( t \) = thickness of shaft wall and

\( A_m \) = the total mean area enclosed by the shaft measured to the midpoint of the wall.

BEAMS

Shearing Force and Bending Moment Sign Conventions

1. The bending moment is positive if it produces bending of the beam concave upward (compression in top fibers and tension in bottom fibers).

2. The shearing force is positive if the right portion of the beam tends to shear downward with respect to the left.

\[ \text{POSITIVE BENDING} \quad \text{NEGATIVE BENDING} \]

\[ \text{POSITIVE SHEAR} \quad \text{NEGATIVE SHEAR} \]

The relationship between the load \( (q) \), shear \( (V) \), and moment \( (M) \) equations are:

\[ q(x) = -\frac{dV(x)}{dx} \]

\[ V = \frac{dM(x)}{dx} \]

\[ V_2 - V_1 = \int_{x_1}^{x_2} \left[-q(x)\right] \, dx \]

\[ M_2 - M_1 = \int_{x_1}^{x_2} V(x) \, dx \]

Stresses in Beams

\[ \varepsilon_y = -\frac{y}{\rho} \]

\( y \) = the distance from the neutral axis to the longitudinal fiber in question.

\[ \rho = \text{the radius of curvature of the deflected axis of the beam, and} \]

\[ \varepsilon_y = \text{the distance from the neutral axis to the longitudinal fiber in question.} \]

Using the stress-strain relationship \( \sigma = E\varepsilon \),

Axial Stress: \( \sigma_x = -\frac{Ey}{\rho} \), where

\( \sigma_x = \) the normal stress of the fiber located \( y \)-distance from the neutral axis.

\( 1/\rho = M/(EI) \), where 
\( M = \) the moment at the section and 
\( I = \) the moment of inertia of the cross-section.

\( \sigma_x = -\frac{My}{I} \), where

\( \sigma_x = \) the normal stress of the fiber located \( y \)-distance from the neutral axis.

\( 1/\rho = M/(EI) \), where 
\( M = \) the moment at the section and 
\( I = \) the moment of inertia of the cross-section.

Let \( S = I/c \): then, \( \sigma_x = \pm M/S \), where

\( S = \) the elastic section modulus of the beam member.

Transverse shear flow: \( q = VQ/I \) and

Transverse shear stress: \( \tau_{xy} = VQ/(lb) \), where

\( q = \) shear flow,
\( \tau_{xy} = \) shear stress on the surface,
\( V = \) shear force at the section,
\( b = \) width or thickness of the cross-section, and
\( Q = A'y' \), where
\( A' = \) area above the layer (or plane) upon which the desired transverse shear stress acts and 
\( y' = \) distance from neutral axis to area centroid.

**Deflection of Beams**

Using \( 1/\rho = M/(EI) \),

\[ EI \frac{d^2y}{dx^2} = M \]

is the differential equation of deflection curve

\[ EI \frac{d^3y}{dx^3} = dM(x)/dx = V \]

\[ EI \frac{d^4y}{dx^4} = dV(x)/dx = -q \]

Determine the deflection curve equation by double integration (apply boundary conditions applicable to the deflection and/or slope).

\[ EI (dy/dx) = \int M(x) \, dx \]

\[ Ely = \int \left[ \int M(x) \, dx \right] \, dx \]

The constants of integration can be determined from the physical geometry of the beam.

**COLUMNS**

For long columns with pinned ends:

Euler’s Formula

\[ P_{cr} = \frac{\pi^2 EI}{\ell^2} \]

\( P_{cr} = \) critical axial loading,
\( \ell = \) unbraced column length.

Substitute \( I = r^2A \):

\[ P_{cr} = \frac{\pi^2 E}{(r/r)^2} \]

\( r = \) radius of gyration and 
\( r/r = \) slenderness ratio for the column.

For further column design theory, see the CIVIL ENGINEERING and MECHANICAL ENGINEERING sections.

**ELASTIC STRAIN ENERGY**

If the strain remains within the elastic limit, the work done during deflection (extension) of a member will be transformed into potential energy and can be recovered.

If the final load is \( P \) and the corresponding elongation of a tension member is \( \delta \), then the total energy \( U \) stored is equal to the work \( W \) done during loading.

\[ U = W = P\delta/2 \]

The strain energy per unit volume is

\[ u = U/AL = \sigma^2/2E \]

(for tension)

**MATERIAL PROPERTIES**

<table>
<thead>
<tr>
<th>Material</th>
<th>Units</th>
<th>Steel</th>
<th>Aluminum</th>
<th>Cast Iron</th>
<th>Wood (Fir)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of Elasticity, ( E )</td>
<td>Mpsi</td>
<td>29.0</td>
<td>10.0</td>
<td>14.5</td>
<td>1.6</td>
</tr>
<tr>
<td></td>
<td>GPa</td>
<td>200.0</td>
<td>69.0</td>
<td>100.0</td>
<td>11.0</td>
</tr>
<tr>
<td>Modulus of Rigidity, ( G )</td>
<td>Mpsi</td>
<td>11.5</td>
<td>3.8</td>
<td>6.0</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>GPa</td>
<td>80.0</td>
<td>26.0</td>
<td>41.4</td>
<td>4.1</td>
</tr>
<tr>
<td>Poisson's Ratio, ( \nu )</td>
<td></td>
<td>0.30</td>
<td>0.33</td>
<td>0.21</td>
<td>0.33</td>
</tr>
<tr>
<td>Coefficient of Thermal Expansion, ( \alpha )</td>
<td>(10^{-6}/\circ F)</td>
<td>6.5</td>
<td>13.1</td>
<td>6.7</td>
<td>1.7</td>
</tr>
<tr>
<td></td>
<td>(10^{-6}/\circ C)</td>
<td>11.7</td>
<td>23.6</td>
<td>12.1</td>
<td>3.0</td>
</tr>
</tbody>
</table>
### Beam Deflection Formulas – Special Cases

(δ is positive downward)

<table>
<thead>
<tr>
<th>Case</th>
<th>Formula</th>
<th>Maximum Deflection</th>
<th>Maximum Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \delta = Pa^2 \left( 3x - a \right) ) for ( x &gt; a )</td>
<td>( \delta_{\text{max}} = \frac{Pa^2}{6EI} \left( 3L - a \right) )</td>
<td>( \phi_{\text{max}} = \frac{Pa^2}{2EI} )</td>
</tr>
<tr>
<td>2.</td>
<td>( \delta = \frac{Px^2}{6EI} \left( -x + 3a \right) ) for ( x \leq a )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>( \delta = \frac{w_0 x^2}{24EI} \left( x^2 + 6L^2 - 4Lx \right) )</td>
<td>( \delta_{\text{max}} = \frac{w_0 L^4}{8EI} )</td>
<td>( \phi_{\text{max}} = \frac{w_0 L^3}{6EI} )</td>
</tr>
<tr>
<td>4.</td>
<td>( \delta = \frac{M_o x^2}{2EI} )</td>
<td>( \delta_{\text{max}} = \frac{M_o L^2}{2EI} )</td>
<td>( \phi_{\text{max}} = \frac{M_o L}{EI} )</td>
</tr>
<tr>
<td>5.</td>
<td>( \delta = \frac{Pb}{6EI} \left[ L \left( x - a \right)^3 - x^3 + \left( L^2 - b^2 \right)x \right] ) for ( x &gt; a )</td>
<td>( \delta_{\text{max}} = \frac{Pb \left( L^2 - b^2 \right)^{3/2}}{9\sqrt{3EI}} ) at ( x = \sqrt{\frac{L^2 - b^2}{3}} )</td>
<td>( \phi_{1} = \frac{Pab(2L - a)}{6EI} ) ( \phi_{2} = \frac{Pab(2L - b)}{6EI} )</td>
</tr>
<tr>
<td>6.</td>
<td>( \delta = \frac{w_0 x}{24EI} \left( L^3 - 2Lx^2 + x^3 \right) )</td>
<td>( \delta_{\text{max}} = \frac{5w_o L^4}{384EI} )</td>
<td>( \phi_{1} = \phi_{2} = \frac{w_o L^3}{24EI} )</td>
</tr>
</tbody>
</table>

*Note:*

\( R_1 = \frac{Pb}{L} \)

\( R_2 = \frac{Pa}{L} \)

\( R_1 = w_0 \frac{L}{2} \)

\( R_2 = w_0 \frac{L}{2} \)

\( M_l = L \frac{W_o L}{2} \)

\( M_r = M_l = w_o \frac{L^2}{12} \)

\( \delta(x) = \frac{W_o}{24EI} \left( L^2 - 2Lx + x^2 \right) \)

\( |\delta_{\text{max}}| = \frac{W_o L^4}{384EI} \) at \( x = \frac{L}{2} \)

\( |\phi_{\text{Max}}| = 0.008 \frac{W_o L^3}{24EI} \) at \( x = \frac{L}{2} \pm \frac{L}{\sqrt{12}} \)

---