CHAPTER 13

DYNAMICS OF ROBOTIC SYSTEMS

13.1 The Lagrangian Equations

Dynamic equations for robotic arms are derived using the Lagrangian equations which are expressed as

$$\frac{d}{dt}\Big(\frac{\partial KE}{\partial \omega_k}\Big) - \frac{\partial KE}{\partial \phi_k} = Q_k \quad \text{for} \quad k = 1, 2, \ldots, n \tag{13.1}$$

where

$\phi_1, \phi_2, \ldots \phi_n$ are the generalized coordinates of the system

Q_k, $k = 1, 2, \ldots, n$ represent the generalized forces, and

KE is the kinetic energy of the system.

The generalized force vector $Q = [Q_1 \;\; Q_2 \; \ldots \; Q_n]^T$ is given by

$$Q = \underline{\tau}^e + \underline{\tau}^a = J^T \underline{F} + \underline{\tau}^a \tag{13.2}$$

where $\underline{F}$ is the external force/torque vector (here, $\underline{\tau}^e = J^T \underline{F}$).

13.2 Derivation of Dynamic Equations

The kinetic energy of the j'th link is written as

$$KE_j = \frac{1}{2}\, I_j \Omega_j^2 + \frac{1}{2}\, m_j[(v_j^x)^2 + (v_j^y)^2]$$

where v_j^x , v_j^y and Ω_j are the absolute velocities of link j which are expressed in terms of the g-functions as

$$\Omega_j = \sum_{i=1}^{j} g_{ji}\, \omega_i$$

$$v_j^x = \sum_{i=1}^{j} g_{ji}^x \, \omega_i$$

$$v_j^y = \sum_{i=1}^{j} g_{ji}^y \, \omega_i$$

where

$$g_{ji}^{\{\cdot\}} = \frac{\partial \{\cdot\}_{cj}}{\partial \phi_i}$$

is the g-function for the center of mass of link j. Substitution of these velocity terms in the kinetic energy expression yields

$$KE_j = \frac{1}{2} \, I_j \Big(\sum_{i=1}^{j} g_{ji} \, \omega_i \Big)^2 + \frac{1}{2} \, m_j \Big[\Big(\sum_{i=1}^{j} \, g_{ji}^x \omega_i \Big)^2 + \Big(\sum_{i=1}^{j} \, g_{ji}^y \omega_i \Big)^2 \Big] \tag{13.3}$$

The kinetic energy of a robotic system is then the sum of the kinetic energies of each individual link, that is,

$$KE = \sum_{j=1}^{n} KE_j \tag{13.4}$$

Equation (13.4) is written in a compact form

$$KE = \frac{1}{2} \, \underline{\omega}^T \, [I^*] \, \underline{\omega} \tag{13.5}$$

where

$$\underline{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix}$$

and I^* is the generalized inertia matrix which is an $n \times n$ matrix consisting of elements I_{ij}^*

$$I^* = [I_{ij}^*]$$

whose entries are defined by

$$I_{ij}^* = \sum_{k=1}^{n} \{ I_k \; g_{ki} \; g_{kj} + m_k \; g_{ki}^x \; g_{kj}^x + m_k \; g_{ki}^y \; g_{kj}^y \}$$

The generalized inertia matrix, I^*, is a symmetric positive definite matrix which implies

$$\underline{\omega}^T \; [I^*] \; \underline{\omega} \; > 0 \quad \text{for all nonzero } \underline{\omega}$$

and

$$I_{ij}^* = I_{ji}^*$$

This property then guarantees that I^* is always invertible. We now evaluate the second term of the Lagrangian by differentiating Equation (13.5) with respect to ϕ_k

$$\frac{\partial KE}{\partial \phi_k} = \frac{\partial}{\partial \phi_k} \{ \frac{1}{2} \; \underline{\omega}^T \; [I^*] \; \underline{\omega} \}$$

$$= \frac{1}{2}\,\underline{\omega}^T\left[\frac{\partial I^*}{\partial \phi_k}\right]\underline{\omega} \tag{13.6}$$

In order to find the first term of the Lagrangian, we will first differentiate the kinetic energy expression with respect to ω_k

$$\frac{\partial KE}{\partial \omega_k} = \frac{1}{2}\,\frac{\partial}{\partial \omega_k}\Big\{ I^*_{11}\omega_1\omega_1 + I^*_{12}\omega_1\omega_2 + \ldots + I^*_{1k}\omega_1\omega_k + \ldots + I^*_{1n}\omega_1\omega_n$$

$$+ I^*_{21}\omega_2\omega_1 + \ldots + I^*_{2k}\omega_2\omega_k + \ldots$$

$$\ldots$$

$$+ I^*_{k1}\omega_k\omega_1 + I^*_{k2}\omega_k\omega_2 + \ldots + I^*_{kk}\omega_k\omega_k + \ldots + I^*_{kn}\omega_k\omega_n$$

$$\ldots$$

$$\ldots + I^*_{nk}\omega_n\omega_k + \ldots + I^*_{nn}\omega_n\omega_n\Big\} = \frac{1}{2}\,2\,\Big\{I^*_{k1}\omega_1 + I^*_{k2}\omega_2 + \ldots + I^*_{kn}\omega_n\Big\}$$

$$= \begin{bmatrix} I^*_{k1} & I^*_{k2} & \ldots & I^*_{kn}\end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2 \\ \vdots \\ \omega_n\end{bmatrix}$$

$$\frac{\partial KE}{\partial \omega_k} = [I^*_k]\,\underline{\omega} \tag{13.7}$$

In Equation (13.7), $[I^*_k]$ denotes the k'th row of I^*. Taking the time derivative of Equation (13.7), we obtain

$$\frac{d}{dt}\Big(\frac{\partial KE}{\partial \omega_k}\Big) = \frac{d}{dt}\Big([I^*_k]\,\underline{\omega}\Big) = [I^*_k]\,\underline{\alpha} + \frac{d[I^*_k]}{dt}\,\underline{\omega} \tag{13.8}$$

Each element of the k'th row of I^* is a function of the joint displacements

$$I^*_{ki} = I^*_{ki}(\underline{\phi}) \quad \text{for} \quad i = 1, 2, \ldots, n$$

Therefore, in order to find time derivative of each of these terms, we apply the chain rule, which yields

$$\frac{dI^*_{ki}}{dt} = \frac{\partial I^*_{ki}}{\partial \phi_1}\omega_1 + \frac{\partial I^*_{ki}}{\partial \phi_2}\omega_2 + \ldots \frac{\partial I^*_{ki}}{\partial \phi_n}\omega_n = \sum_{j=1}^{n}\frac{\partial I^*_{ki}}{\partial \phi_j}\omega_j \quad \text{for} \quad i = 1, 2, \ldots, n$$

By carrying out the above differentiation for n elements of the k'th row of I^*, we obtain

$$\frac{d[I^*_k]}{dt} = \left[\sum_{j=1}^{n}\frac{\partial I^*_{k1}}{\partial \phi_j}\omega_j \quad \sum_{j=1}^{n}\frac{\partial I^*_{k2}}{\partial \phi_j}\omega_j \quad \ldots \quad \sum_{j=1}^{n}\frac{\partial I^*_{kn}}{\partial \phi_j}\omega_j\right]$$

which is rewritten as

$$\frac{d[I^*_k]}{dt} = \begin{bmatrix}\omega_1 & \omega_2 & \ldots & \omega_n\end{bmatrix}\begin{bmatrix} \frac{\partial I^*_{k1}}{\partial \phi_1} & \frac{\partial I^*_{k2}}{\partial \phi_1} & \ldots & \frac{\partial I^*_{kn}}{\partial \phi_1} \\ \frac{\partial I^*_{k1}}{\partial \phi_2} & \frac{\partial I^*_{k2}}{\partial \phi_2} & \ldots & \frac{\partial I^*_{kn}}{\partial \phi_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial I^*_{k1}}{\partial \phi_n} & \frac{\partial I^*_{k2}}{\partial \phi_n} & \ldots & \frac{\partial I^*_{kn}}{\partial \phi_n}\end{bmatrix} \tag{13.9}$$

which is simply represented by

$$\frac{d[I_k^*]}{dt} = \underline{\omega}^T \left[\frac{\partial [I_k^*]}{\partial \underline{\phi}}\right] \tag{13.10}$$

We now write the Lagrangian equation for the dynamic system as the sum of the terms from Equations (13.2), (13.6), (13.8) and (13.10):

$$[I_k^*]\ \underline{\alpha} + \underline{\omega}^T \left[\frac{\partial [I_k^*]}{\partial \underline{\phi}}\right] \underline{\omega} - \frac{1}{2}\ \underline{\omega}^T \left[\frac{\partial I^*}{\partial \phi_k}\right] \underline{\omega} = \tau_k^e + \tau_k^a \tag{13.11}$$

for $k = 1, 2, \ldots, n$. In the above formulation the first term represents the inertial loading due to tangential accelerations, while the second and third terms include the inertial effects due to centrifugal and Coriolis acceleration components. We combine these velocity-related terms (second and third terms) and rewrite Equation (13.11) as follows:

$$[I_k^*]\ \underline{\alpha} = \underline{\omega}^T P_k^*\, \underline{\omega} + \tau_k^e + \tau_k^a\,, \quad \text{for} \quad k = 1, 2, \ldots, n$$

where the $n \times n$ matrix $P_k^* = P_k^*(\underline{\phi})$ is defined by

$$P_k^* = \frac{1}{2}\left[\frac{\partial I^*}{\partial \phi_k}\right] - \left[\frac{\partial [I_k^*]}{\partial \underline{\phi}}\right]$$

Finally, n dynamic equations for an n-link arm is expressed in vector format:

$$[I^*]\ \underline{\alpha} = \underline{b}(\underline{\phi}, \underline{\omega}) + \underline{\tau}^e + \underline{\tau}^a \tag{13.12}$$

where $\underline{\tau}^e$ represents the effect of external loads as seen at joints, and the n-dimensional vector $\underline{b}(\underline{\phi}, \underline{\omega})$ is given by

$$\underline{b} = \underline{b}(\underline{\phi}, \underline{\omega}) = \begin{bmatrix} \underline{\omega}^T\ P_1^*\ \underline{\omega} \\ \underline{\omega}^T\ P_2^*\ \underline{\omega} \\ \vdots \\ \underline{\omega}^T\ P_n^*\ \underline{\omega} \end{bmatrix}$$

If the robot's movement is slow, and $\underline{\omega}$ and $\underline{\alpha}$ are negligible, or the robot's mass content is negligible, the above set of dynamic equations in Equation (13.12) reduces to

$$\underline{\tau}^a = -\underline{\tau}^e = -\ J^T\ \underline{F}$$

which is the static force equilibrium equation derived earlier.

13.3 Forward and Inverse Dynamics Problems

When the joint trajectories and external forces acting on the robot are described a priori, and the necessary input actuator torques are required, we have the forward dynamics problem:

$$\text{Given}: \ \underline{\phi}(t),\ \underline{\omega}(t),\ \underline{\alpha}(t),\ \underline{F}(t)$$

$$\text{Find}: \ \underline{\tau}^a(t)$$

The solution to this problem is determined by rewriting Equation (13.12) as

$$\underline{\tau}^a = \underline{\tau}^a(t) = [I^*]\ \underline{\alpha} - \ \underline{b}(\underline{\phi}, \underline{\omega}) - \underline{\tau}^e$$

Note that although all parameters in the above equation are time dependent, we usually do not show that dependency explicitly in our notation for clarity. The inverse problem assumes that the initial joint positions and velocities are given. If, in addition, input joint torques are also described, then the resulting robot motion can be determined from the dynamic equations:

$$[I^*]\ \underline{\alpha} = \ \underline{b}(\underline{\phi}, \underline{\omega}) + \underline{\tau}^e + \underline{\tau}^a \qquad (13.13)$$

In this case, the known and unknown parameters for the inverse dynamics problem are summarized as follows:

$$\text{Given}: \ \underline{\phi}(t_0),\ \underline{\omega}(t_0),\ \ \underline{\tau}^a(t)$$

$$\text{Find}: \ \underline{\phi}(t),\ \underline{\omega}(t) \ \ \text{for} \ \ t > t_0$$

The solution to this problem is determined by integrating Equation (13.13) which is a system of n second-order differential equations. For this purpose, these equations are rewritten as a set of $2n$ first-order differential equations in the state-space form

$$\dot{\underline{\phi}} = \underline{\omega}$$

$$\dot{\underline{\omega}} = [I^*]^{-1}\{\ \underline{b}(\underline{\phi}, \underline{\omega}) + \underline{\tau}^a + \underline{\tau}^e\}$$

which are numerically integrated since these form a system of coupled, nonlinear differential equations.

13.4 Derivation of Dynamic Equations for A 2R Arm

13.4.1 Kinetic Energy Expression

Consider the 2-DOF planar arm consisting of two revolute joints as shown in Figure 13.1. The necessary g-functions are found as follows:

$$\Phi_1 = \phi_1 \Rightarrow g_{11} = \frac{\partial \Phi_1}{\partial \phi_1} = 1$$

$$g_{12} = \frac{\partial \Phi_1}{\partial \phi_2} = 0$$

$$\Phi_2 = \phi_1 + \phi_2 \Rightarrow g_{21} = \frac{\partial \Phi_2}{\partial \phi_1} = 1$$

$$\Rightarrow g_{22} = \frac{\partial \Phi_2}{\partial \phi_2} = 1$$

$$x_1 = x_{C_1} = l_1 \cos\phi_1 \Rightarrow g^x_{11} = \frac{\partial x_1}{\partial \phi_1} = -l_1 \sin\phi_1$$

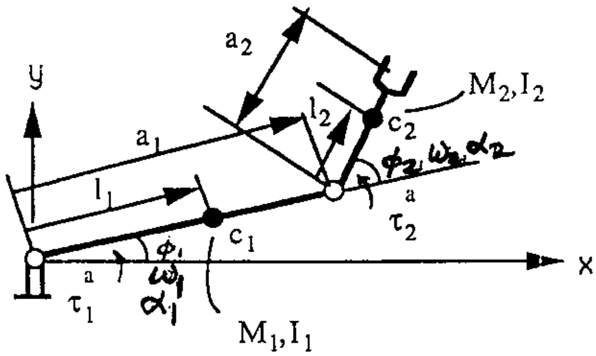


Figure 13.1 2R planar arm

$$g_{12}^{x} = \frac{\partial x_1}{\partial \phi_2} = 0$$

$$x_2 = x_{C_2} = a_1 \cos\phi_1 + l_2 \cos(\phi_1 + \phi_2) \Rightarrow g_{21}^{x} = -a_1 \sin\phi_1 - l_2 \sin(\phi_1 + \phi_2)$$

$$g_{22}^{x} = -l_2 \sin(\phi_1 + \phi_2)$$

$$y_1 = y_{C_1} = l_1 \sin\phi_1 \Rightarrow g_{11}^{y} = \frac{\partial y_1}{\partial \phi_1} = l_1 \cos\phi_1$$

$$g_{12}^{y} = \frac{\partial y_1}{\partial \phi_2} = 0$$

$$y_2 = y_{C_2} = a_1 \sin\phi_1 + l_2 \sin(\phi_1 + \phi_2) \Rightarrow g_{21}^{y} = a_1 \cos\phi_1 + l_2 \cos(\phi_1 + \phi_2)$$

$$g_{22}^{y} = l_2 \cos(\phi_1 + \phi_2)$$

The kinetic energy of the arm is written as

$$KE = \frac{1}{2}\Big[I_1\Omega_1^2 + m_1\{(v_1^x)^2 + (v_1^y)^2\}\Big] + \frac{1}{2}\Big[I_2\Omega_2^2 + m_2\{(v_2^x)^2 + (v_2^y)^2\}\Big]$$

where

$$\Omega_1 = g_{11}\omega_1, \qquad \Omega_2 = g_{21}\omega_1 + g_{22}\omega_2$$

$$v_1^x = g_{11}^x\omega_1, \qquad v_2^x = g_{21}^x\omega_1 + g_{22}^x\omega_2$$

$$v_1^y = g_{11}^y\omega_1, \qquad v_2^y = g_{21}^y\omega_1 + g_{22}^y\omega_2$$

If these expressions are then substituted into the kinetic energy expression we obtain the following form

$$KE = \frac{1}{2}\Big[I_1 g_{11}^2 + m_1\{(g_{11}^x)^2 + (g_{11}^y)^2\}\Big]\omega_1^2 + \frac{1}{2}I_2\Big[g_{21}\omega_1 + g_{22}\omega_2\Big]^2$$

$$+\frac{1}{2}m_2(g_{21}^x\omega_1+g_{22}^x\omega_2)^2+\frac{1}{2}m_2(g_{21}^y\omega_1+g_{22}^y\omega_2)^2$$

The terms in this equation are expanded and regrouped as

$$KE=\frac{1}{2}\Big\{(I_1g_{11}^2+m_1(g_{11}^x)^2+m_1(g_{11}^y)^2+I_2g_{21}^2+m_2(g_{21}^x)^2+m_2(g_{21}^y)^2)\omega_1^2$$

$$+2(I_2g_{21}g_{22}+m_2g_{21}^xg_{22}^x+m_2g_{21}^yg_{22}^y)\omega_1\omega_2+(I_2g_{22}^2+m_2(g_{22}^x)^2+m_2(g_{22}^y)^2)\omega_2^2\Big\} \tag{13.14}$$

which is a quadratic equation in terms of the angular velocities ω_1 and ω_2. We can thus view Equation (13.14) in an equivalent form:

$$KE=\frac{1}{2}\Big\{I_{11}^*\omega_1^2+2I_{12}^*\omega_1\omega_2+I_{22}^*\omega_2^2\Big\} \tag{13.15}$$

If we now define

$$I_{12}^*=I_{21}^*$$

then, Equation (13.15) can be represented in a matrix-vector format:

$$KE=\frac{1}{2}\begin{bmatrix}\omega_1 & \omega_2\end{bmatrix}\begin{bmatrix}I_{11}^* & I_{12}^*\\ I_{21}^* & I_{22}^*\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}=\frac{1}{2}\,\underline{\omega}^T[I^*]\,\underline{\omega}$$

Comparing terms in Equations (13.14) and (13.15), one concludes that

$$I_{11}^*=(I_1g_{11}^2+m_1(g_{11}^x)^2+m_1(g_{11}^y)^2+I_2g_{21}^2+m_2(g_{21}^x)^2+m_2(g_{21}^y)^2$$

$$I_{12}^*=I_{21}^*=I_2g_{21}g_{22}+m_2g_{21}^xg_{22}^x+m_2g_{21}^yg_{22}^y \tag{13.16}$$

and

$$I_{22}^*=I_2g_{22}^2+m_2(g_{22}^x)^2+m_2(g_{22}^y)^2$$

Note that we could have started from the earlier definition of I^* and directly substituted in the formula to derive the entries I_{ij}^*, that is,

$$I_{ij}^*=\sum_{k=1}^{n}[I_k\ g_{ki}\ g_{kj}+m_k\ g_{ki}^x\ g_{kj}^x+m_k\ g_{ki}^y\ g_{kj}^y]$$

Noting that in this case $n=2$, the above formula yields the same results as in Equation (13.16). We now substitute the values of the g-functions evaluated earlier and derive explicit representations for the terms in Equation (13.16):

$$I_{11}^*=I_1+m_1l_1^2(\sin^2\phi_1+\cos^2\phi_1)+I_2$$

$$+m_2\Big[a_1^2\sin^2\phi_1+l_2^2\sin^2(\phi_1+\phi_2)+2a_1l_2\sin\phi_1\sin(\phi_1+\phi_2)$$

$$+a_1^2\cos^2\phi_1+l_2^2\cos^2(\phi_1+\phi_2)+2a_1l_2\cos\phi_1\cos(\phi_1+\phi_2)\Big]$$

or

$$I_{11}^*=I_1+m_1l_1^2+I_2+m_2(a_1^2+l_2^2+2a_1l_2\cos\phi_2)$$

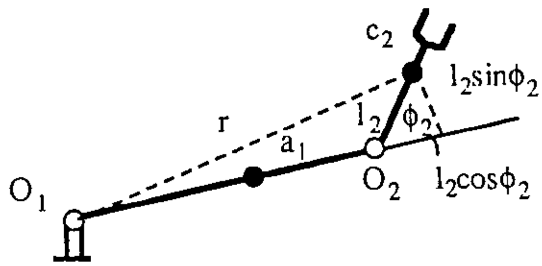


Figure 13.2 Geometry of the 2R arm

Note that as shown in Figure 13.2, r is given by

$$r = \sqrt{a_1^2 + l_2^2 + 2a_1 l_2 \cos\phi_2}$$

I_{11}^* has the following significance:

$$I_{11}^* = I_1\Big|_{O_1} + I_2\Big|_{O_1}$$

where

$$I_1\Big|_{O_1} = I_1 + m_1 l_1^2$$

$$I_2\Big|_{O_1} = I_2 + m_2(a_1^2 + l_2^2 + 2a_1 l_2 \cos\phi_2)$$

is the sum of the equivalent link inertias seen at O_1. Similarly, $I_{12}^*(= I_{21}^*)$ and I_{22}^* expressions are derived as

$$I_{12}^* = I_{21}^* = I_2 + m_2(a_1 l_2 \cos\phi_2) + m_2 l_2^2$$

and

$$I_{22}^* = I_2 + m_2 l_2^2$$

13.4.2 Velocity-Related Terms

Having the I^* matrix derived, recall that we are trying to express the dynamic equations in the form

$$I^*\underline{\alpha} = \underline{b}(\underline{\phi}, \underline{\omega}) + \underline{\tau}^e + \underline{\tau}^a$$

In order to evaluate the velocity-related terms in $\underline{b}(\underline{\phi}, \underline{\omega})$, we need to carry out the partial differentiation of I^* with respect to joint displacements:

$$\frac{\partial I^*}{\partial \phi_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{\partial I^*}{\partial \phi_2} = \begin{bmatrix} -2m_2 a_1 l_2 \sin\phi_2 & -m_2 a_1 l_2 \sin\phi_2 \\ -m_2 a_1 l_2 \sin\phi_2 & 0 \end{bmatrix}$$

$$\frac{\partial I_1^*}{\partial \underline{\phi}} = \begin{bmatrix} 0 & 0 \\ -2m_2 a_1 l_2 \sin\phi_2 & -m_2 a_1 l_2 \sin\phi_2 \end{bmatrix}$$

$$\frac{\partial I_2^*}{\partial \underline{\phi}} = \begin{bmatrix} 0 & 0 \\ -m_2 a_1 l_2 \sin\phi_2 & 0 \end{bmatrix}$$

Hence, the vector $\underline{b}$

$$\underline{b} = \begin{bmatrix} \vdots \\ \vdots \\ \underline{\omega}^T \left(\frac{1}{2} \left[\frac{\partial I^*}{\partial \phi_k} \right] - \left[\frac{\partial I_k^*}{\partial \underline{\phi}} \right] \right) \underline{\omega} \\ \vdots \\ \vdots \end{bmatrix}$$

will be

$$\underline{b} = \begin{bmatrix} \underline{\omega}^T \left(\frac{1}{2} \left[\frac{\partial I^*}{\partial \phi_1} \right] - \left[\frac{\partial I_1^*}{\partial \underline{\phi}} \right] \right) \underline{\omega} \\ \underline{\omega}^T \left(\frac{1}{2} \left[\frac{\partial I^*}{\partial \phi_2} \right] - \left[\frac{\partial I_2^*}{\partial \underline{\phi}} \right] \right) \underline{\omega} \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} \underline{\omega}^T \left[\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -2m_2 a_1 l_2 \sin\phi_2 & -m_2 a_1 l_2 \sin\phi_2 \end{pmatrix} \right] \underline{\omega} \\ \underline{\omega}^T \left[\frac{1}{2} \begin{pmatrix} -2m_2 a_1 l_2 \sin\phi_2 & -m_2 a_1 l_2 \sin\phi_2 \\ -m_2 a_1 l_2 \sin\phi_2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -m_2 a_1 l_2 \sin\phi_2 & 0 \end{pmatrix} \right] \underline{\omega} \end{bmatrix}$$

or

$$\underline{b} = \begin{bmatrix} 2m_2 a_1 l_2 \sin\phi_2 \omega_1 \omega_2 + m_2 a_1 l_2 \sin\phi_2 \omega_2^2 \\ -m_2 a_1 l_2 \sin\phi_2 \omega_1^2 \end{bmatrix} \tag{13.17}$$

13.4.3 Dynamic Equations

Finally, the results derived above give the dynamic equations for the 2R as follows:

Dynamic equation for link 1

$$\left[I_1 + m_1 l_1^2 + I_2 + m_2 \left(a_1^2 + l_2^2 + 2a_1 l_2 \cos\phi_2 \right) \right] \alpha_1 + \left[I_2 + m_2 (a_1 l_2 \cos\phi_2) + m_2 l_2^2 \right] \alpha_2$$
$$= 2m_2 a_1 l_2 \sin\phi_2 \omega_1 \omega_2 + m_2 a_1 l_2 \sin\phi_2 \omega_2^2 + \tau_1^e + \tau_1^a \tag{13.18}$$

Dynamic equation for link 2

$$\left[I_2 + m_2 (a_1 l_2 \cos\phi_2) + m_2 l_2^2 \right] \alpha_1 + \left[I_2 + m_2 l_2^2 \right] \alpha_2 = -m_2 a_1 l_2 \sin\phi_2 \omega_1^2 + \tau_2^e + \tau_2^a \tag{13.19}$$

These equations describe the link motion and represent dynamic equilibrium conditions. The vector $\underline{\tau}^e$ describes the effect of external forces and is given by $\underline{\tau}^e = J^T \underline{F}$, where the specific structure of the Jacobian, J, depends on the arm design and the external force vector $\underline{F}$ as covered previously. $\underline{\tau}^a$ is the set of input actuator torques. Note again that the dynamic equations form a set of coupled, nonlinear, second-order ordinary differential equations.

13.4.4 Physical Significance of Dynamic Equations

Now let us consider the physical significance of the terms $I^* \underline{\alpha}$ and $\underline{b}(\underline{\phi}, \underline{\omega})$. First consider the tangential acceleration components acting at the center of gravity of each link. In Figure 13.3, the tangential acceleration of C_1 is $\mathbf{a}^t_{C_1}$ with a magnitude $l_1 \alpha_1$ and a direction as shown in the figure.

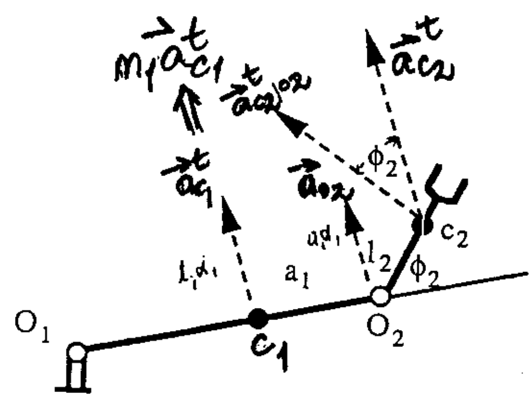


Figure 13.3 Tangential components of acceleration

The tangential acceleration for C_2 is expressed as

$$\mathbf{a}^t_{C_2} = \mathbf{a}^t_{O_2} + \mathbf{a}^t_{C_2/O_2}$$

where

$$|\mathbf{a}^t_{O_2}| = a_1 \alpha_1, \text{ and } |\mathbf{a}^t_{C_2/O_2}| = l_2(\alpha_1 + \alpha_2)$$

These acceleration components will create inertial forces $m_1 \mathbf{a}^t_{C_1}$, $m_2 \mathbf{a}^t_{O_2}$ and $m_2 \mathbf{a}^t_{C_2/O_2}$, which, in turn, will induce the following torque about the second joint O_2:

$$M^t_{O_2} = m_2 \underbrace{a_1 \alpha_1 \cos \phi_2} \underbrace{l_2}_{moment\ arm} + m_2 l_2 (\alpha_1 + \alpha_2) \underbrace{l_2}_{moment\ arm}$$

where the term in the large brace is the component of $\mathbf{a}^t_{O_2}$ which creates a torque about O_2. If we also add the inertial effect due to rotational acceleration of the link (that is, $\alpha_1 + \alpha_2$ for link 2), $M^t_{O_2}$ becomes

$$m_2 a_1 \alpha_1 \cos \phi_2 l_2 + m_2 l_2 (\alpha_1 + \alpha_2) l_2 + I_2 (\alpha_1 + \alpha_2)$$

$$= \Big[I_2 + m_2(l_2^2 + a_1 l_2 \cos\phi_2)\Big]\alpha_1 + \Big[I_2 + m_2 l_2^2\Big]\alpha_2 = I_{21}^*\alpha_1 + I_{22}^*\alpha_2$$

The inertia forces $m_1\mathbf{a}_{C_1}^t$, $m_1\mathbf{a}_{O_2}^t$, $m_1\mathbf{a}_{C_2/O_2}^t$, and the rotational inertia of link 1, $I_1\alpha_1$, and that of link 2, $I_2(\alpha_1+\alpha_2)$, create a moment M_{O1}^t about point O_1:

$$M_{O_1}^t = M_{O_2}^t + m_2 a_1^2\alpha_1 + m_2 l_2(\alpha_1+\alpha_2)\cos\phi_2 a_1 + m_1 l_1^2\alpha_1 + I_1\alpha_1 + I_2(\alpha_1+\alpha_2)$$

$$= \Big[I_1 + m_1 l_1^2 + I_2 + m_2(l_2^2 + a_1^2 + 2a_1 l_2\cos\phi_2)\Big]\alpha_1 + \Big[I_2 + m_2(l_2^2 + a_1 l_2\cos\phi_2)\Big]\alpha_2$$

$$= I_{11}^*\alpha_1 + I_{12}^*\alpha_2$$

The above exercise demonstrates that $I^*\underline{\alpha}$ does actually represent the inertial torques (or forces) due to the tangential and rotational acceleration components.

Finally, we check the effect of centrifugal acceleration on the dynamic equations. C_1 and C_2 have centrifugal accelerations $\mathbf{a}_{C_1}^c$ and $\mathbf{a}_{C_2}^c = \mathbf{a}_{O_2}^c + \mathbf{a}_{C_2/O_2}^c$ as shown in Figure 13.4.

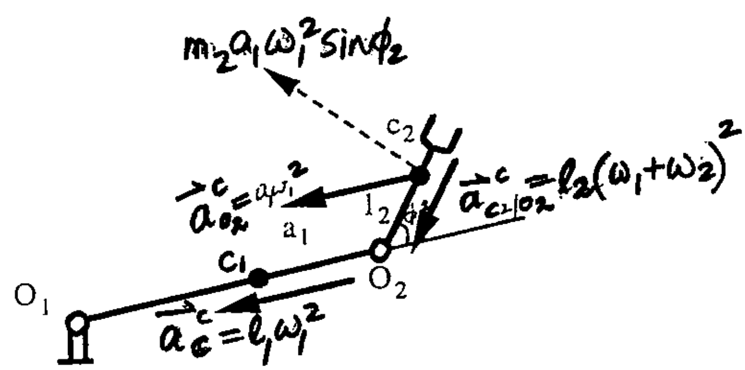


Figure 13.4 Centrifugal components of acceleration

The moment created by these inertial forces about O_2 is

$$M_{O_2}^c = \underbrace{m_2 a_1\omega_1^2 \sin\phi_2}\, l_2 = -b_2$$

where the grouped term is the effective component of the this moment about O_2. Similarly the moment about O_1 is given by

$$M_{O_1}^c = M_{O_2}^c - m_2 l_2(\omega_1+\omega_2)^2 \sin\phi_2 a_1$$

The equivalent system is shown in Figure 13.5.

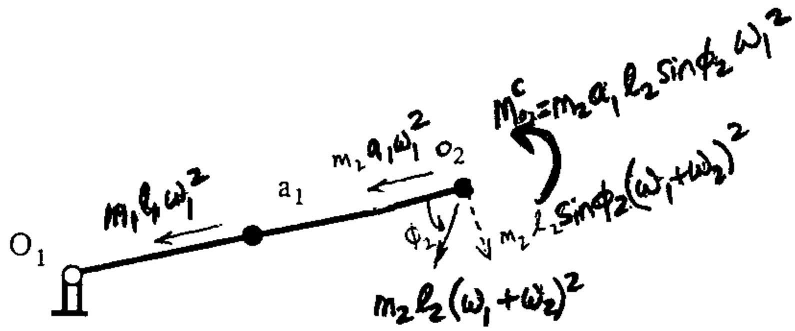


Figure 13.5 Equivalent force representation

$$M^c_{O_1} = m_2 a_1 l_2 \sin\phi_2 \omega_1^2 - m_2 a_1 l_2 \sin\phi - 2(\omega_1 + \omega_2)^2$$
$$= m_2 a_1 l_2 \sin\phi - 2(\omega_1^2 - \omega_1^2 - 2\omega_1\omega_2 - \omega_2^2)$$
$$M^c_{O_1} = -m_2 a_1 l_2 \sin\phi_2(\omega_1\omega_2 + \omega_2^2) = -b_1$$

Hence, $-\underline{b}(\underline{\phi}, \underline{\omega})$ represents the inertial torque due to the centrifugal acceleration component.

Remark

Dynamic equations are derived in the form

$$I^* \underline{\alpha} = \underline{b} + \underline{\tau}^e + \underline{\tau}^a$$

while the inertial effects are represented by

$$I^* \underline{\alpha} - \underline{b} = \underline{\tau}^e + \underline{\tau}^a$$

according to the Lagrangian derivation. Hence, in our notation $-\underline{b}$ represents the inertial torque rather than $\underline{b}$.

Example 1

Consider the two-link 2R arm as shown in Figure 13.1. Assume that the arm constant parameters are given by

$$m_1 = 10 \text{ kg}, \quad I_1 = 4 \text{ kg.m}^2, \quad a_1 = 0.8 \text{ m}, \quad l_1 = 0.4 \text{ m}$$

$$m_2 = 20 \text{ kg}, \quad I_2 = 9 \text{ kg.m}^2, \quad a_2 = 0.5 \text{ m}, \quad l_1 = 0.3 \text{ m}$$

If instantaneous joint displacements, velocities and accelerations are required to be

$$\phi_1 = 30°, \;\; \omega_1 = 15 \text{ sec}^{-1}, \;\; \alpha_1 = -20 \text{ sec}^2$$

$$\phi_2 = -10°, \;\; \omega_2 = 5 \text{ sec}^{-1}, \;\; \alpha_2 = 8 \text{ sec}^2$$

find the instantaneous kinetic energy of this manipulator.

First let us derive the instantaneous g-functions:

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{21} = 1, \quad g_{22} = 1$$

$$g_{11}^x = -l_1 \sin\phi_1 = -0.4 \sin 30° = -0.2 \text{ m}$$

$$g_{12}^x = 0$$

$$g_{21}^x = -a_1 \sin\phi_1 - l_2 \sin(\phi_1 + \phi_2) = -0.8 \sin 30° - 0.3 \sin 20° = -0.503 \text{ m}$$

$$g_{22}^x = -l_2 \sin(\phi_1 + \phi_2) = -0.3 \sin 20° = -0.103 \text{ m}$$

$$g_{11}^y = l_1 \cos\phi_1 = 0.4 \cos 30° = 0.346 \text{ m}$$

$$g_{12}^y = 0$$

$$g_{21}^y = a_1 \cos\phi_1 + l_2 \cos(\phi_1 + \phi_2) = 0.8 \cos 30° + 0.3 \cos 20° = 0.975 \text{ m}$$

$$g_{22}^y = l_2 \cos(\phi_1 + \phi_2) = 0.3 \cos 20° = 0.282 \text{ m}$$

Using these g-functions, the entries of the generalized inertia matrix I^* are derived as follows:

$$I_{11}^* = (I_1 g_{11}^2 + m_1 (g_{11}^x)^2 + m_1 (g_{11}^y)^2 + I_2 g_{21}^2 + m_2 (g_{21}^x)^2 + m_2 (g_{21}^y)^2)$$

$$= 4(1)^2 + 10(-0.2)^2 + 10(0.346)^2 + 9(1)^2 + 20(-0.503)^2 + 20(0.975)^2 = 38.67 \text{ kg.m}^2$$

$$I_{12}^* = I_{21}^* = I_2 g_{21} g_{22} + m_2 g_{21}^x g_{22}^x + m_2 g_{21}^y g_{22}^y$$

$$= 9(1)(1) + 20(-0.503)(-0.103) + 20(0.975)(0.282) = 15.54 \text{ kg.m}^2$$

and

$$I_{22}^* = I_2 g_{22}^2 + m_2 (g_{22}^x)^2 + m_2 (g_{22}^y)^2$$

$$= 9(1)^2 + 20(-0.103)^2 + 20(0.282)^2 = 10.80 \text{ kg.m}^2$$

Hence, the instantaneous kinetic energy of the arm is calculated from

$$KE = \frac{1}{2} \, \underline{\omega}^T I^* \underline{\omega}$$

as

$$KE = \frac{1}{2} [15 \quad 5] \begin{bmatrix} 38.67 & 15.54 \\ 15.54 & 10.80 \end{bmatrix} \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

$$= \frac{1}{2} [38.67(15)^2 + 2(15.54)(15)(5) + 10.80(5)^2] = 5650.88 \text{ N.m}$$

Example 2

For the robot described in the above example, find the necessary input actuator torques.

Since the generalized inertia matrix I^* is already found, we will calculate the $\underline{b}$ and $\underline{T}^e$ vectors as follows:

$$\frac{\partial I^*}{\partial \phi_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{\partial I^*}{\partial \phi_2} = \begin{bmatrix} -2m_2 a_1 l_2 \sin\phi_2 & -m_2 a_1 l_2 \sin\phi_2 \\ -m_2 a_1 l_2 \sin\phi_2 & 0 \end{bmatrix} = \begin{bmatrix} 1.67 & 0.83 \\ 0.83 & 0 \end{bmatrix}$$

$$\frac{\partial I_1^*}{\partial \underline{\phi}} = \begin{bmatrix} 0 & 0 \\ -2m_2 a_1 l_2 \sin\phi_2 & -m_2 a_1 l_2 \sin\phi_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1.67 & 0.83 \end{bmatrix}$$

$$\frac{\partial I_2^*}{\partial \underline{\phi}} = \begin{bmatrix} 0 & 0 \\ -m_2 a_1 l_2 \sin\phi_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0.83 & 0 \end{bmatrix}$$

The vector $\underline{b}$ can now be determined:

$$\underline{b} = \begin{bmatrix} \underline{\omega}^T \left[\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1.67 & 0.83 \end{pmatrix} \right] \underline{\omega} \\ \underline{\omega}^T \left[\begin{pmatrix} 0.83 & 0.42 \\ 0.42 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0.83 & 0 \end{pmatrix} \right] \underline{\omega} \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} [15 \quad 5] \begin{bmatrix} 0 & 0 \\ -1.67 & -0.83 \end{bmatrix} \begin{bmatrix} 15 \\ 5 \end{bmatrix} \\ [15 \quad 5] \begin{bmatrix} 0.83 & 0.42 \\ -0.42 & 0 \end{bmatrix} \begin{bmatrix} 15 \\ 5 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -1.67(15)(5) - 0.83(5)^2 \\ 0.83(15)^2 + 0.42(15)(5) - 0.42(15)(5) + 0(5)^2 \end{bmatrix} = \begin{bmatrix} -146 \\ 186.75 \end{bmatrix} \text{ N.m}$$

Note that in this case we could have calculated $\underline{b}$ from Equation (13.17). In order to determine the $\underline{T}^e$ vector, let us assume that only link weights $W_1 = 100$ N and $W_2 = 200$ N (acceleration due to gravity = 10 ms^{-2}) are considered as external forces:

$$\underline{T}^e = J^T \underline{F} = \begin{bmatrix} g_{11}^y & g_{21}^y \\ 0 & g_{22}^y \end{bmatrix} \begin{bmatrix} -W_1 \\ -W_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.346 & 0.975 \\ 0 & 0.282 \end{bmatrix} \begin{bmatrix} -100 \\ -200 \end{bmatrix} = \begin{bmatrix} -229.6 \\ -56.4 \end{bmatrix} \text{ N.m}$$

Since the dynamic equations are given by

$$I^* \underline{\alpha} = \underline{b} + \underline{T}^e + \underline{T}^a$$

the input actuator torques are determined from

$$\underline{\tau}^a = I^* \underline{\alpha} - \underline{b} - \underline{\tau}^e$$

$$= \begin{bmatrix} 38.67 & 15.54 \\ 15.54 & 10.80 \end{bmatrix} \begin{bmatrix} -20 \\ 8 \end{bmatrix} - \begin{bmatrix} 146 \\ 186.75 \end{bmatrix} - \begin{bmatrix} -229.6 \\ -56.4 \end{bmatrix}$$

$$\underline{\tau}^a = \begin{bmatrix} -273.48 \\ -354.75 \end{bmatrix} \text{ N.m}$$

13.5 Control of Robots: Computed-Torque Method

Robots can be operated by applying the input actuator torques calculated in the forward dynamics problem. However, this assumes perfect system modeling, perfect knowledge of system parameters such as the mass content, link lengths, etc. This implies that such an approach will lack robustness. In order to improve system reliability, usually feedback loops are introduced in control schemes in addition to the actuator torques. Here we will briefly introduce one such approach commonly called the computed-torque method.

Suppose that we want the robot to have a particular joint position, velocity and acceleration (obtained by trajectory planning). We affix a subscript d to the position, velocity and acceleration variables to denote these 'desired' values. We can then write an error term defined by the position error

$$\underline{e} = \underline{\phi}_d - \underline{\phi}$$

the velocity error

$$\underline{\dot{e}} = \underline{\omega}_d - \underline{\omega}$$

and the acceleration error

$$\underline{\ddot{e}} = \underline{\alpha}_d - \underline{\alpha}$$

where $\underline{\phi}$, $\underline{\omega}$ and $\underline{\alpha}$ now represent the 'actual' kinematic parameters. By using the computed-torque method we calculate the driving torque required at the actuators as

$$\underline{\tau}^a = -\,\underline{b}(\underline{\phi},\underline{\omega}) - \underline{\tau}^e + [I^*]\,(\underline{\alpha}_d + K_p\,\underline{e} + K_v\,\underline{\dot{e}}) \tag{13.20}$$

where K_p and K_v are the gain matrices which are of dimension $n \times n$ for an n-link manipulator. The simplest choice of K_p and K_v is diagonal matrices where the error in each link is fed back to the actuator corresponding to that link. The effect of having these diagonal gain matrices will become apparent when we consider the closed-loop response of such a control system. To evaluate the closed-loop response, we substitute the value of the computed actuator torque in Equation (13.20):

$$[I^*]\,\underline{\alpha} = \underline{b}(\underline{\phi},\underline{\omega}) + \underline{\tau}^e - \underline{b}(\underline{\phi},\underline{\omega}) - \underline{\tau}^e + [I^*]\,(\underline{\alpha}_d + K_p\,\underline{e} + K_v\,\underline{\dot{e}})$$

or

$$\underline{\alpha} = \underline{\alpha}_d + K_p\,\underline{e} + K_v\,\underline{\dot{e}}$$

which then yields

$$\underline{\ddot{e}} + K_v\,\underline{\dot{e}} + K_p\,\underline{e} = 0 \tag{13.21}$$

This is the error-driven dynamic equations of the system or the closed-loop system dynamics. Now if K_v and K_p are diagonal matrices (with positive gains for stability), then this set of equations will decouple and result in n second-order equations. These equations have solutions that are characteristic unforced harmonic responses of second-order systems which decay exponentially and thus, in this case, the error will be eliminated. Figure 13.6 shows the block diagram representation of the computed-torque method, while the schematic in Figure 13.7 depicts the realization of this controller with an interface to a computer system.

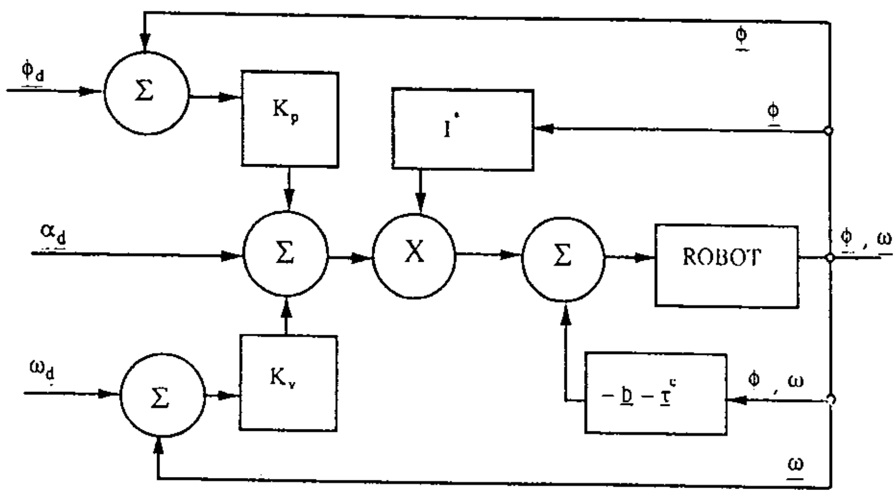
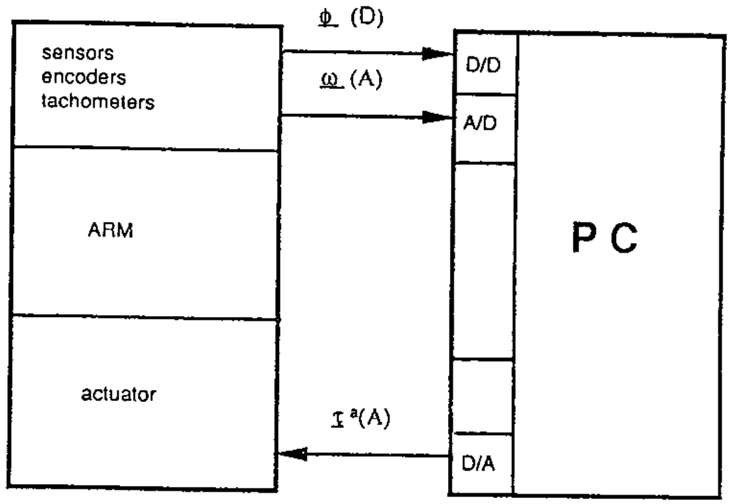


Figure 13.6 Block diagram for the computed-torque method



controller tasks

Inverse kinematics :

Given : Position,velocity,acceleration of end-effector.

Find : Desired joint positions,velocities, accelerations.

Get present position,velocity of arm from encoders and sensors. Find position error and velocity error.

Calculate g, h, I^*, P^*, J,$J^T\underline{F}$, $\underline{b}$

Calculate the actuator torque .

Figure 13.7 Computer interface to a robot

Problem Set

1. For the planar arm shown in Figure P1, find the position and orientation of the end-effector in terms of the coordinate frame shown. Can the position and the orientation be specified independently for this RP robot?

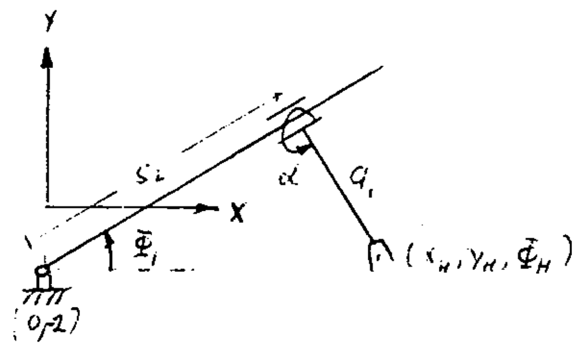


Figure P1

2. Consider the RRP manipulator shown in Figure P2.
   (a) What is the mobility of this manipulator; that is, how many degrees of freedom does it possess?
   (b) Can the position and orientation be randomly specified and achieved in practice through the positioning of joints?
   (c) If the joint variables are given by

   $$\Phi_1 = 14°, \quad \Phi_2 = 38°, \quad s_3 = 1.03 \text{ m}$$

   and the link and joint constant variables are

   $$a_1 = 3.2 \text{ m}, \quad a = 1.6 \text{ m}, \quad \beta = 30°$$

   calculate the position and orientation of the end-effector (x_H, y_H and Φ_H). Is this solution unique?

3. Which geometric structure of a planar manipulator yields the most straightforward solution to the inverse position analysis problem (that is, 3R, PRP, etc.)? Can you find a geometry that will yield a unique solution to the inverse position analysis problem?

4. For the planar robot shown in Figure P4, find expressions for the relative joint positions that will position the end-effector at (x_H, y_H and Φ_H). For the values of the hand position and orientation specified below, calculate the joint displacements:

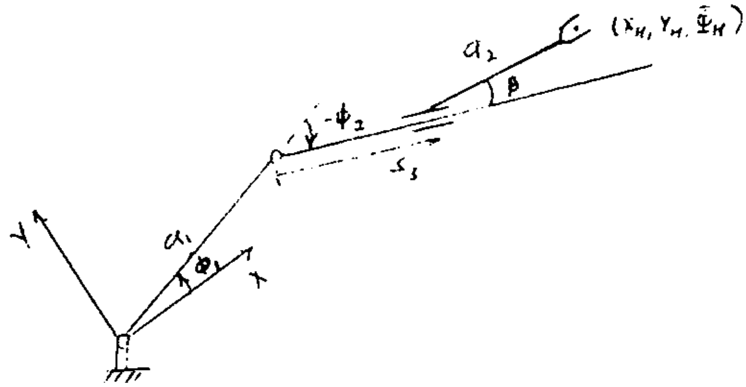


Figure P2

(a) $x_H = 0.1$ m, $y_H = 0.75$ m and $\Phi_H = 180°$.
(b) $x_H = 0.75$ m, $y_H = 0.1$ m and $\Phi_H = 180°$.
(c) Can the inverse position solution be guaranteed if $L_3 > L_2$?

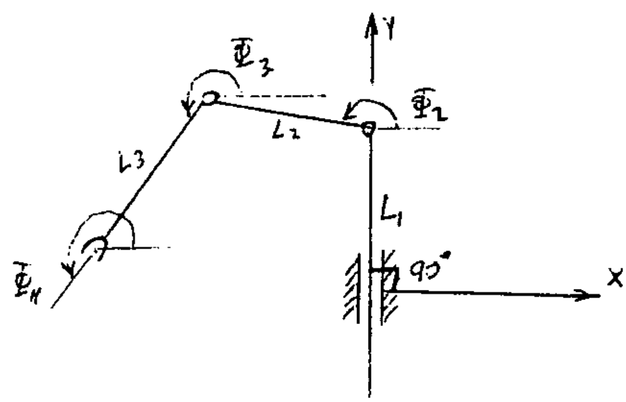


Figure P4

5. Find the joint displacements ϕ_1 and ϕ_2 for the 2R manipulator shown in Figure P5 if the following hand parameters are given:
(a) $x_H = 0.23$ m, $y_H = 0.76$ m.
(b) $x_H = 0.35$ m, $\Phi_H = 30°$. Also find the corresponding y_H for this configuration.
(c) What do multiple solutions in each case mean geometrically?

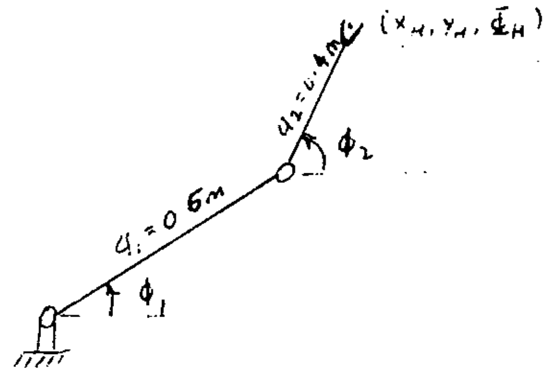


Figure P5

6. For the RP manipulator in Figure P6, end-effector position and constant arm parameters are measured to be

$$x_H = 0.866 \text{ m}, \qquad y_H = 1.5 \text{ m}, \qquad a = 1 \text{ m}, \qquad \beta = 60^\circ$$

Find the necessary joint positions ϕ_1 and s_2 to maintain this configuration. Is the solution unique?

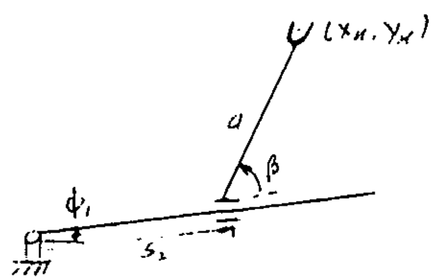


Figure P6

7. Consider the 3R planar manipulator shown in Figure P7. If the hand position and orientation are specified as

$$x_H = 3 \text{ m}, \qquad y_H = 7 \text{ m}, \qquad \Phi_H = 100^\circ$$

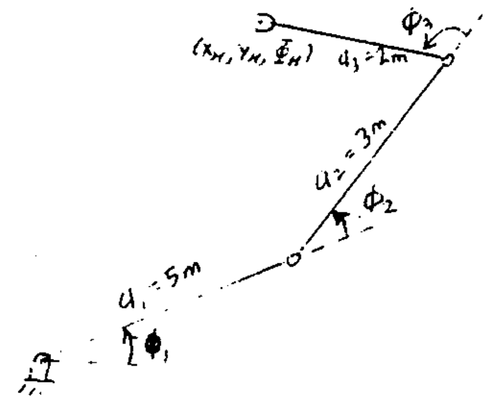


Figure P7

find the joint positions ϕ_1, ϕ_2 and ϕ_3 by adding a hypothetical link and instantaneously forming a four-bar mechanism. Is this solution unique?

8. For the manipulator shown in Figure P8,
   (a) Derive the g- and h-functions (rotational and translational components) in terms of the joint displacements ϕ_1, ϕ_2 and ϕ_3 for points A and H.
   (b) Write the Jacobian matrices for these points and derive the singularity conditions.
   (c) Illustrate these singularity configurations.

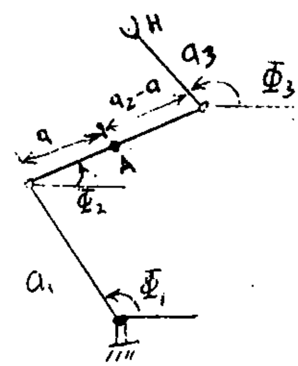


Figure P8

9. As shown in Figure P9 by dashed line, a 2R manipulator is required to move along a straight line with a constant linear velocity of 0.3 m/s. Assuming that the manipulator has

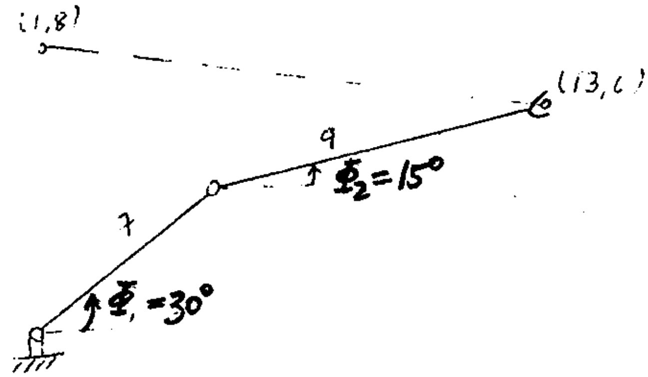


Figure P9

an initial configuration as shown in the figure, find the required joint velocities to achieve this motion. (Hint: You may break up the path into 3 to 4 sub-intervals and perform the calculations at these discrete points.) Plot the velocity of each joint against its displacement. Are the joint velocities linear? Why is it so? Find the joint accelerations at these points and plot them against the joint displacements. Are the joint accelerations linear? Why?

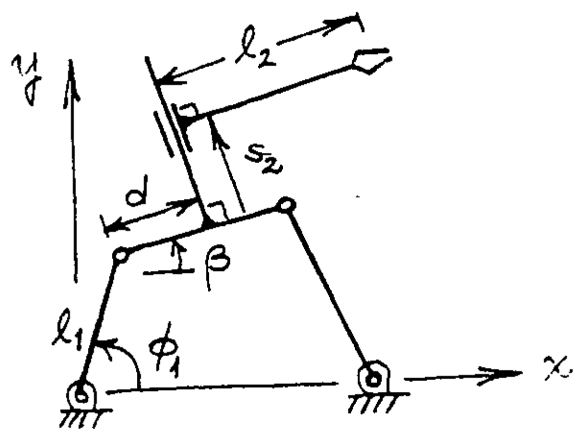


Figure P10

10. A simple hybrid robot is shown in Figure P10. If β is approximated by

$$\beta = 1.3\phi_1 + \phi_1^2$$

find expressions for the g- and h-functions of the end-effector, and evaluate the x component of the end-effector velocity.

11. The following numerical data are given for a two-link arm:

$$J = \begin{bmatrix} g_{H_1}^x & g_{H_2}^x \\ g_{H_1}^y & g_{H_2}^y \\ g_{H_1} & g_{H_2} \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -3 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{units}$$

$$[h_H^x] = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$[h_H^y] = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$[h_H] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$V_H^x = 15 \text{ m/s}, \qquad V_H^y = -25 \text{ m/s}$$

$$\alpha_1 = 30 \text{ s}^{-2}, \qquad \alpha_2 = -15 \text{ s}^{-2}$$

(a) Calculate the joint velocities ω_1 and ω_2.
(b) Find the end-effector acceleration a_H^x and a_H^y.

12. For the 2-DOF RP manipulator shown in Figure P12, calculate the required actuator torques to keep the system in static equilibrium.

$$m_1 = 20 \text{ kg}, \quad m_2 = 10 \text{ kg}, \quad m_3 = 40 \text{ kg}, \quad g = 9.8 \text{ m/s}^2$$

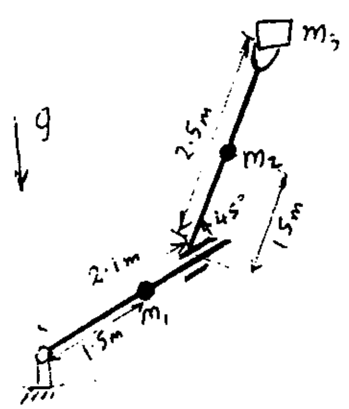


Figure P12

13. Consider a two-link 2R planar arm with $m_1 = 5$ kg, $m_2 = 10$ kg, $I_1 = 2$ kg.m^2 and $I_2 = 4$ kg.m^2. At a given position the g-functions for the centers of gravity of each link are given by

$$g_{11}^x = -2 \text{ m}, \quad g_{11}^y = 3 \text{ m}, \quad g_{11} = 1$$

$$g_{12}^x = 0 \text{ m}, \quad g_{12}^y = 0 \text{ m}, \quad g_{12} = 0$$

$$g_{21}^x = -4 \text{ m}, \quad g_{21}^y = 3 \text{ m}, \quad g_{21} = 1$$

$$g_{22}^x = -2 \text{ m}, \quad g_{22}^y = 0 \text{ m}, \quad g_{22} = 1$$

(a) Find the instantaneous generalized inertia matrix I^* for this robot in the given position.
(b) Calculate the kinetic energy of the arm for $\omega_1 = -2 \text{ s}^{-1}$ and $\omega_2 = 1 \text{ s}^{-1}$.

14. Given that the kinetic energy of a link j is

$$KE_j = \frac{1}{2} I_j \Omega^2 + \frac{1}{2} M_j \{(v_j^x)^2 + (v_j^y)^2\}$$

(a) Show that the total kinetic energy of an n-link manipulator may be represented by

$$KE = \frac{1}{2} \underline{\omega}^T [I^*] \underline{\omega}$$

(b) For the PR arm shown in Figure P14, express the I^* entries in terms of the g-functions.
(c) Derive the complete dynamic equations for this arm.

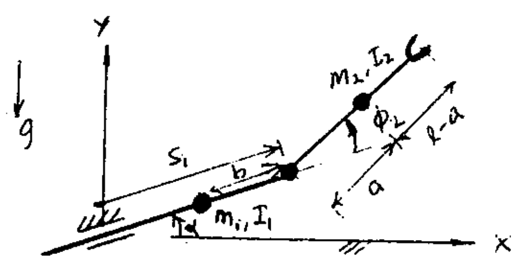


Figure P14